Kaluza-Klein Branes

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Preprint DAMTP-2001-42

21 May 2001

Abstract

We examine Kaluza-Klein branes in detail. Specifically, we show that codimension four submanifolds that are stationary under a semi-free circle action may be interpreted as branes or antibranes in the Kaluza-Klein reduced space that are magnetically charged under the Kaluza-Klein field strength. We derive the equation in cohomology that is satisfied by such a brane using an explicit construction of the Thom class of the normal bundle of the brane worldvolume in the reduced space. This may be applied to both the D6-brane of Type IIA String Theory, and also to various recent constructions of magnetic branes immersed in fluxbrane backgrounds. We then go on to study the special case of monopole-antimonopole production in a five-dimensional Kaluza-Klein theory, illustrating our arguments with various concrete examples.

1 Introduction

In [1], various examples of magnetically charged strings or p-branes were constructed, at the level of the appropriate low-energy effective theory, where the gauge field derives from Kaluza-Klein reduction on a circle. More recently, various papers have given further examples of this construction [2], where spherical, or more generally, tubular, branes are immersed (in the physical rather than strict mathematical sense) in a background magnetic fluxbrane; that is, the brane solution approaches the fluxbrane solution asymptotically. The fluxbrane is essentially just a generalisation of the Melvin Universe [3] and provides the magnetic force required to prevent the brane from collapsing due to its own tension. Such solutions are of course typically unstable.

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In this paper, we focus on the case of Kaluza-Klein branes. By definition, these are branes that are magnetically charged under a $U(1)$ Kaluza-Klein field strength. In [1], various examples of Kaluza-Klein branes were constructed by considering a circle action on some manifold X, and then Kaluza-Klein reducing on the circle direction; the branes are identified with the stationary points of the circle action. This construction produces both the Type IIA D6-brane, at the level of supergravity, and also various examples of magnetic p-branes immersed in fluxbrane backgrounds. The aim of the present paper is to make the relation between stationary points of circle actions and Kaluza-Klein branes more precise. In general, it is not obvious how such stationary point sets may be interpreted as branes in the reduced space, and in particular, why the corresponding branes may be charged under the Kaluza-Klein field strength. The above facts were deduced in [1], and the barrage of recent papers [2], by examining specific examples, rather than giving a general argument. It is also worth noting that in all these examples, and indeed in general, the dilaton diverges as one approaches the brane. Hence, physically, the space is decompactifying near the brane, and one should therefore work in the higher dimensional spacetime in a neighbourhood of the brane. However, in order to examine properties of the brane from the point of view of the base spacetime, one needs to interpret the brane as an object that is intrinsic to the base. We fill this gap in the literature. Assuming that the higher dimensional spacetime X takes the form $X = \mathbb{R} \times M$ where $\mathbb{R}$ is the time direction and M is the spatial manifold on which we dimensionally reduce, we show that codimension four (with respect to M) submanifolds that are stationary under a semi-free circle action may be interpreted as branes or antibranes in the reduced space that are magnetically charged with respect to the Kaluza-Klein field strength, G_2. Such a brane acts as a source for G_2

$$[dG_2] = [\delta(W)] \tag{1.1}$$

This holds[1] as an equation in the cohomology group $H^3(B)$ of the spatial base B, where $\delta(W)$ is a closed three-form that is Poincaré dual to the brane worldvolume W in B, and has support on W. We derive this equation from the Kaluza-Klein perspective using an explicit construction of the Thom class of the normal bundle of W in B. The present paper therefore both formalises and generalises previous work, placing the examples of [1], together with the examples contained in more recent papers, in a general setting. Note that the reduced spacetime is actually of the form $\mathbb{R} \times B$. The time direction is topologically trivial, and in particular will not enter into our topological considerations. We therefore simply neglect the factor of $\mathbb{R}$ in most of the paper, dealing either with the Riemannian manifold M, which is a spacelike hypersurface in X, or the Kaluza-Klein reduction B of M.

We then go on to use the above ideas in the context of monopole-antimonopole production in a five-dimensional Kaluza-Klein theory. In dimension four, Kaluza-Klein monopoles are 0-branes which are charged under the Kaluza-Klein $U(1)$ gauge field. The details of monopole-antimonopole production are constrained by the topology of the nucleation surface. In particular, the fact that the total number of monopoles and antimonopoles must be equal (charge conservation), and that the total number of defects produced is given by the Euler characteristic of the nucleation surface, may be derived using various G-index theorems. Using a result of Fintushel on the classification of circle actions on simply-connected four-manifolds, together

[1]For an antibrane there is an extra minus sign. We have normalised G_2 such that the brane charge is 1. More generally, one has $[dG_2] = \pm Q[\delta(W)]$ for a brane (antibrane) of charge Q $(-Q)$.

with some standard cobordism results, we are able to classify completely the possible topologies of the nucleation surface, assuming the latter is simply connected. Finally, we give several examples which describe the nucleation of Kaluza-Klein monopoles and antimonopoles either in the presence of a positive cosmological constant, or a thin domain wall.

The plan of the paper is as follows. In Section 2, we motivate the present paper by reminding the reader of some of the examples of Kaluza-Klein branes contained in reference [1]. In Section 3, we analyse circle actions in detail. This section is somewhat technical, and is included only to correct some of the misconceptions in the current literature. In Section 4, we derive equation (1.1) from two points of view. Firstly, we assume that we have a codimension three brane W living on a d-dimensional manifold B that couples magnetically to G_2 via a Wess-Zumino coupling

$$(-1)^D \int_{\mathbb{R}\times W} C_{D-4} \tag{1.2}$$

in the 'string frame' action, where C_{D-4} is the potential for the dual field strength of G_2 and we also define $\dim(X) = D$ and $\dim(M) = d+1$, so that $d = D-2$. The equation (1.1) is then derived by a constrained variation of the total action. This is more or less standard. The second point of view is to regard W as arising, in a way described more precisely later, from a codimension four (again, with respect to M) stationary point set of a semi-free circle action of the higher dimensional space M. The Kaluza-Klein 2-form G_2 is not defined on the whole base B. We first show that there exists an extension of G_2 on the whole space B such that equation (1.1) is satisfied; this is related to a specific construction of the Thom class of the normal bundle of W in B. We then show that this extension is unique, as far as cohomology is concerned. We also give a detailed account of brane charge, illustrating the discussion using some of the examples in Section 2.

As mentioned, in Section 5 we study the case $D = 5$ in detail. Kaluza-Klein branes are simply monopoles in this case, and we use various theorems on circle actions in order to deduce the qualitative details of monopole-antimonopole pair production. Finally, in Section 6 we give several explicit examples, illustrating the ideas of the previous section. Our conclusions are contained in Section 7.

2 Motivation

In this section, we briefly remind the reader of some of the examples of Kaluza-Klein branes constructed in [1]. These will serve both as motivation for the present paper, and also as illustrations of some of the more abstract topological ideas we shall encounter later.

2.1 The Kaluza-Klein monopole

The Kaluza-Klein monopole is a solution to the canonical five-dimensional Kaluza-Klein theory. The monopole itself is a 0-brane that is magnetically charged under the $U(1)$ Kaluza-Klein gauge field. Specifically, the Ricci-flat five-manifold X is given by a metric product $X = \mathbb{R}\times M$ where $\mathbb{R}$ is the time direction, and M is the Euclidean (anti-)self-dual Taub-NUT metric. Thus

$$ds^2 = -dt^2 + ds^2_{\mathrm{Taub-NUT}} \tag{2.1}$$

where

$$ds^2_{\mathrm{Taub-NUT}} = \left(\frac{r+a}{r-a}\right) dr^2 + 4a^2 \left(\frac{r-a}{r+a}\right) (d\psi + \cos\theta d\phi)^2 + (r^2 - a^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2}$$

The Taub-NUT metric is hyperKähler, with holonomy $SU(2)$, and is therefore Ricci-flat. The radial coordinate r takes the range $a \le r < \infty$, and (ψ, θ, ϕ) are Euler angles on S^3. The manifold is topologically $\mathbb{R}^4$ and asymptotically flat.

The monopole solution (2.1) admits a circle isometry[2] generated by the Killing vector field $\partial/\partial\psi$. This has a one-dimensional stationary point set given by $\{r = a, -\infty < t < \infty\}$, which is interpreted as the monopole worldline. We shall return to this example frequently during the rest of the paper.

2.2 Magnetic spherical p-branes immersed in fluxbranes

The obvious way to construct higher dimensional p-brane solutions is to take the product of (2.1) with p flat spatial directions; these will be magnetic p-brane solutions of a $(p+5)$-dimensional Kaluza-Klein theory. However, an alternative construction was presented in [1], resulting in a spherical p-brane worldvolume. This has recently been generalised to tubular branes [2]. We describe the case of a magnetically charged spherical p-brane in a $(p+5)$-dimensional Kaluza-Klein theory. Asymptotically these solutions all approach the fluxbrane solutions described in references [1] and [2].

The solution is again a metric product $X = \mathbb{R} \times M$ with $\mathbb{R}$ a trivial time direction, but now the manifold M is the $(p+4)$-dimensional Euclidean Schwarzschild solution

$$ds^2 = -dt^2 + \left(1 - \left(\frac{r_H}{r}\right)^{p+1}\right) d\tau^2 + \left(1 - \left(\frac{r_H}{r}\right)^{p+1}\right)^{-1} dr^2 + r^2 d\Omega_{p+2} \tag{2.3}$$

where $d\Omega_n$ is the metric on the unit n-sphere. We write the metric on S^{p+2} as

$$d\Omega_{p+2} = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\Omega_p \tag{2.4}$$

and take the circle action generated by the Killing vector field

$$\frac{\partial}{\partial\tau} + \frac{1}{R}\frac{\partial}{\partial\phi} \tag{2.5}$$

where $R = \frac{2r_H}{p+1}$, and then Kaluza-Klein reduce on the circle. The stationary points of the circle action are given by $\{r = r_H, \theta = 0, -\infty < t < \infty\}$, which is interpreted as the p-brane worldvolume. The spatial section is the sphere S^p, and by analogy with the monopole solution one can see that the p-brane in the base is magnetically charged under the Kaluza-Klein gauge field. The details may be found in the original reference [1]. The magnetic p-brane solution asymptotically approaches a fluxbrane solution at large radius.

[2]We shall use the terms circle action and $U(1)$ action interchangeably.

2.3 The Type IIA D6-brane

Our final example is the D6-brane of Type IIA String Theory. This is magnetically charged[3] under the Kaluza-Klein two-form that derives from the $D = 11$ metric. In fact, we have already covered this example in our comments in the last section. One may construct, at the level of supergravity, a BPS D6-brane. Specifically, the M-Theory solution is

$$ds^2 = -dt^2 + dx_1^2 + \ldots + dx_6^2 + ds^2_{\mathrm{Taub-NUT}} \tag{2.6}$$

Since the Taub-NUT solution is hyperKähler, it admits two independent parallel spinors[4]. The D6-brane solution (2.6) therefore breaks half of the supersymmetries of the M-Theory vacuum.

3 Kaluza-Klein circle reduction

In this section, we describe Kaluza-Klein reduction on a circle, where the circle direction is given by a smooth $U(1)$ action. This action must be *semi-free* if we are to avoid orbifold singularities in the reduced space. This is largely misunderstood in the physics literature, so we give a very careful treatment. We illustrate the general discussion throughout with concrete examples.

We briefly remind the reader of some definitions regarding group actions on manifolds [16]. A smooth action $\Phi : G \times M \to M$ of the group G on M is said to be free if given any $p \in M$ such that $\Phi(g, p) = p$, then g is the identity element $e \in G$. A point $p \in M$ is said to be fixed if there is some non-trivial $g \in G$, $g \neq e$, such that $\Phi(g, p) = p$. Thus we may say that the action of Φ is free if it has no fixed points.

A point $p \in M$ is said to be stationary if $\Phi(g, p) = p$, $\forall g \in G$. We specifically make this distinction between fixed points and stationary points as it will be important below; often in the literature one finds that no such distinction is made. The action of Φ is said to be semi-free if all fixed points are in fact stationary points. It follows that the isotropy groups $G_p \subset G$ are either all of G, or the trivial group, $\forall p \in M$. Finally, the action is said to be effective if each $g \neq e$ in G moves at least one point in M; that is, if $\Phi(g, p) = p$, $\forall p \in M$, then $g = e$. We tacitly assume that our group actions are effective in this paper.

3.1 Stationary points of circle actions

The total spacetime manifold is $X = \mathbb{R} \times M$, of dimension D, and is typically a trivial metric product, although the Riemannian metric g on M could in principle depend on t. In the rest of this section, we deal only with the space M; that is, the circle action on the factor of $\mathbb{R}$ is always trivial, and thus we may neglect it.

Let (M, g) be an oriented Riemannian manifold of dimension $(d + 1)$, admitting a smooth orientation-preserving isometric circle action $\Phi_\tau : M \to M$, where τ parameterises the $U(1)$ group[5]. Let $M^{U(1)}$ denote the set of stationary points. Then each connected component of

[3]D-brane charge should properly be understood in terms of K-Theory [27]. Provided one is not interested in subtleties such as the precise integrality conditions satisfied by the charges, the use of cohomology is perfectly adequate. In particular, it is sufficient for our purposes.

[4]covariantly constant sections of the spin bundle.

[5]In fact, $\Phi : U(1) \times M \to M$ is smooth if and only if Φ_τ is smooth for each τ [7].

$M^{U(1)}$ constitutes a closed oriented totally geodesic submanifold of M of even codimension. Let F be such a component, of codimension $2r$, and consider the induced action of $U(1)$ on the tangent space T_pM, where $p \in F$. T_pM is a real $U(1)$-module, and hence we may decompose the $U(1)$ action into its irreducible real representations, which, since $U(1)$ is cyclic, are either of the form ± 1 or $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Since F is stationary, the action of Φ_τ on T_pF is trivial, and hence we see that the action on the normal space N_pF of F in M may be decomposed into the product of r commuting 2×2 rotations in r orthogonal 2-planes. If k denotes the Killing vector field associated with Φ_τ, with some normalisation, then ∇k is a 2-form as a consequence of Killing's equation. With respect to an orthonormal frame at p, $\nabla_b k_a$ is therefore skew-symmetric, and is an element of the Lie algebra of $SO(d+1)$. Hence with respect to the orthonormal frame, the $U(1)$ action may be written as the direct sum $1_{d+1-2r} \oplus \bigoplus_{j=1}^{r} R(\kappa_j \tau)$ where κ_j are the skew eigenvalues of $\nabla_b k_a$, and the symbol 1_{d+1-2r} denotes the trivial action on T_pF. N_pF then has a canonical complex structure in which the $U(1)$ action at the tangent space level acts as $e^{i\kappa_j\tau} \in \mathbb{C}$ in the $\mathrm{j^{th}}$ 2-plane, which we identify with $\mathbb{C}$. Since the action is periodic, it follows that the skew eigenvalues $\{\kappa_j \mid j = 1, \ldots, r\}$ must be rationally related, the integers $\{n_j \in \mathbb{Z} \mid j = 1, \ldots, r\}$ relating the eigenvalues determining the number of rotations in each orthogonal $2-$plane in N_pF induced by a single orbit of the $U(1)$ group. Since we require the action to be effective, the $\{n_j \mid j = 1, \ldots, r\}$ necessarily have no common factor. Defining canonical complex coordinates $z_1, \ldots, z_r$ on the normal space N_pF, the action of $\Phi_{\tau *}$ is $z_j \to e^{in_j\tau} z_j$ for each j, where we have taken τ to have the canonical period 2π, and so $\kappa_j = n_j$. Of course, this discussion is independent of the choice of point $p \in F$. Hence, for a generic connected stationary point set F, the circle action canonically decomposes the normal bundle $NF = \cup_{p\in F} N_pF$ of F in M into the sum of r complex line bundles, the induced action on NF being characterised by r integers with highest common factor 1.

To illustrate this discussion, let us consider the isolated stationary point $\{r = a\}$ of the Taub-NUT metric (2.2). Locally, one may choose coordinates in a neighbourhood of this point such that the metric looks like the flat metric on $\mathbb{R}^4$, and the circle action generated by $k = \partial/\partial\psi$ becomes the action $z_j \to e^{in_j\tau} z_j$ where $n_1 = n_2 = (\pm)1$ and $\{z_j \mid j = 1, 2\}$ are complex coordinates on $\mathbb{C}^2 = \mathbb{R}^4$. Thus we see that the circle action on Taub-NUT, generated by the Killing vector field k, is semi-free. The surfaces of constant $r > a$ are topologically three-spheres, and the restriction $k\mid_{S^3}$ generates the Hopf action on S^3, with projection $\mathcal{H} : S^3 \to S^2$. Alternatively, one may take the circle action generated by $-k$. In this case $n_1 = -n_2 = (\pm)1$, the resulting action on the three-spheres is the antiHopf action, and one now has an antimonopole, rather than a monopole.

For an example of a non-semi-free action, simply take $\mathbb{C}^2$ with the action $z_j \to e^{in_j\tau} z_j$ with at least one of the $n_j \neq \pm 1$.

3.2 Circle reduction

We wish to perform a Kaluza-Klein circle reduction on the orbits of k. In order to do this, one must form the quotient space $M/U(1)$. Since the orbits completely degenerate on the stationary points, it is clearly desirable to remove them before taking the quotient. We will later interpret these geodesic submanifolds as topological defects on the base space $M/U(1)$. Now, if M' is a manifold equipped with a smooth effective action of a compact Lie group G with finite isotropy groups G_p at each $p \in M'$, then the quotient space M'/G has the canonical structure of an orbifold [8]. An orbifold is a generalisation of the orbit space of a smooth effective finite group action on a manifold[6]. More specifically, an orbifold is a topological space that can be covered by open sets U_i homeomorphic to $\tilde{U}_i/\Gamma_i$ where the Γ_i are finite groups acting smoothly and effectively on $\tilde{U}_i$, open in $\mathbb{R}^n$. In the case at hand, since the only proper subgroups of $U(1)$ are finite (of the form $\mathbb{Z}_m$), we see that the quotient space $M'/U(1)$ is an orbifold, where $M' = M - M^{U(1)}$. This fact seems to have gone unnoticed in the physics literature.

Since we would like the base space to be a manifold, it follows that we should only consider semi-free U(1) actions; that is, U(1) actions whose only fixed points are stationary points. Then the isotropy groups of M' are all trivial, and the base does indeed inherit a genuine manifold structure. We then have a $U(1)$ principal bundle[7]

$$\pi : M' \to B'$$

Such bundles are classified by their first Chern class $c_1(M') \in H^2(B';\mathbb{Z})$; that is, $U(1)$ bundles over B' are, up to isomorphism, in 1-1 correspondence with elements of the second singular cohomology group of B' with coefficients in $\mathbb{Z}$. One way to see this[8] is to note that the classification of G-bundles over B' depends on the homotopy groups $\pi_n(G)$ of G, and these are all trivial for $n \geq 2$ in the case that $G = U(1)$, and $\pi_1(U(1)) \cong \mathbb{Z}$. Since B' is orientable, by Poincaré duality, $H^2(B';\mathbb{Z}) \cong H_{d-2}(B', \partial B';\mathbb{Z})$ where the latter denotes the $(d-2)$ relative homology group of the pair $(B', \partial B')$ and $\dim(B') = d$. Let S be a codimension two submanifold of B' whose image $[S] \in H_{d-2}(B', \partial B';\mathbb{Z})$ is dual to $c_1(M')$. Then S is a Dirac string, whose lift to M' is referred to as a Misner string in [5]. These codimension two submanifolds are rather heuristically described in the physics literature as submanifolds on which the foliation by surfaces of constant τ breaks down, due to non-trivial twisting of the $U(1)$ bundle. The term string is perhaps somewhat of a misnomer; only when $d = 3$ does one actually obtain curves of real dimension one. We see that any codimension two submanifold S defines a $U(1)$ bundle up to isomorphism, and, conversely, any $U(1)$ bundle defines S up to homology. To see that S is indeed an obstruction to triviality, suppose that S represents the first Chern class of the bundle $\pi : M' \to B'$. Then if $U \subset B'$ is open, the first Chern class of the restriction $M' \mid_U$ is represented by the submanifold $S \cap U$ of U. Applying this fact to $U = B' - S$ implies that the first Chern class of $M' \mid_{B'-S}$ is represented by $0 \in H^2(B' - S;\mathbb{Z})$, and therefore the restriction $\pi : M' \mid_{B'-S} \to B' - S$ is trivial. The singular (co)homology theory is perhaps less familiar to physicists than the de Rham theory, but the use of the singular theory was crucial in our derivation above. We have $H^2(B';\mathbb{R}) \cong H^2(B';\mathbb{Z}) \bigotimes_{\mathbb{Z}} \mathbb{R} \cong H^2_{\mathrm{dR}}(B')$ and so the de Rham cohomology only measures the free part of $H^2(B';\mathbb{Z})$. The de Rham theory is therefore too crude to classify $U(1)$ bundles in general, although in many cases of interest the torsion (the finite part of $H^2(B';\mathbb{Z})$) vanishes and the two approaches are equivalent.

Let us briefly turn back to the Taub-NUT instanton (2.2) again to illustrate these abstract points. Taub-NUT is an example of a space containing a Misner string. Removing the nut $\{r = a\}$ yields a manifold of topology $\mathbb{R}^4 - \{\mathrm{pt}\}$. Dividing out by the free circle action generated by $\partial/\partial\psi$ yields a manifold diffeomorphic to $\mathbb{R}^3 - \{\mathrm{pt}\}$. The two-sphere S^2 is therefore a deformation

[6] Note that this *is* the definition of an orbifold used by physicists.

[7] We denote by M' both the bundle and the total space.

[8] See [25] for a particularly nice account of these ideas.

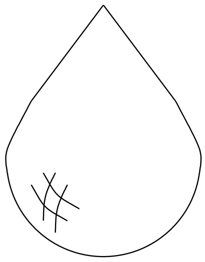

Figure 1: Thurston's teardrop, $\mathbb{C}P^{[1,p]}$. A neighbourhood of the north pole is diffeomorphic to $\mathbb{R}^2/\mathbb{Z}_p$ whereas the south pole is regular.

retraction of this base, and the first Chern class of the $U(1)$ bundle is easily seen to be $1 \in \mathbb{Z} \cong H^2(S^2;\mathbb{Z}) \cong H^2(\mathbb{R}^3 - \{\text{pt}\};\mathbb{Z})$. The Poincaré dual to the Kaluza-Klein two-form $G_2 = \frac{1}{4\pi}\sin\theta d\theta\wedge d\phi \in \Omega^2(\mathbb{R}^3 - \{\text{pt}\})$ may be taken to be the ray $\{\theta = 0\}$. This lifts to the two-manifold $\{\theta = 0\}$ in the total space, which is therefore by definition a Misner string of Taub-NUT space. As we proved earlier, deleting this string trivialises the bundle. This may be seen explicitly here. Deleting the ray $\{\theta = 0\}$ from the base B' is equivalent to deleting a point from the two-sphere S^2 that is a deformation retraction of B'. This leaves us with $\mathbb{R}^2$. But $H^2(\mathbb{R}^2;\mathbb{Z}) \cong 0$ and so by the classification theorem, the bundle must be trivial.

After this slight digression, let us now look at the removal of the stationary points in a little more detail. Let F be as above, and let NF^ϵ denote the open disc bundle of radius $\epsilon > 0$. This is simply defined as the space of all vectors in NF of length at most ϵ. If F is compact[9], by the tubular neighbourhood theorem one may find an $\epsilon > 0$ such that the exponential map maps NF^ϵ equivariantly[10] into a tubular neighbourhood of F in M. The frontier of this neighbourhood is thus a sphere-bundle over F. How does this boundary reduce under the $U(1)$ action? Let $p \in F$ and consider the normal space N_pF. Define the sphere of radius ϵ in $\mathbb{C}^r$ as $S^{2r-1}_\epsilon = \{\{z_1,\ldots,z_r\} \in \mathbb{C}^r \mid \sum_{j=1}^r |z_j|^2 = \epsilon^2\}$. Under the action of $\Phi_{\tau *}$ we have $z_j \to e^{in_j\tau}z_j$ on $\mathbb{C}^r$. Projecting out by this action yields a *weighted projective space*, denoted $\mathbb{C}P^{[n_1,\ldots,n_r]}$, and is a complex orbifold, of complex dimension $r-1$, for general $\{n_j\}$. These spaces are not uncommon in the physics literature[11]. Indeed, in [9], a large number of Calabi-Yau 3-folds were constructed by resolving various hypersurfaces in $\mathbb{C}P^{[n_1,\ldots,n_5]}$. One may characterise the orbifold points as follows. Let $[z_1,\ldots,z_r] \in \mathbb{C}P^{[n_1,\ldots,n_r]}$ and let $m = \text{hcf}\{n_j \mid z_j \neq 0\}$. The points with $m > 1$ correspond to orbifold points, with group $\Gamma = \mathbb{Z}_m$. The set of regular points X_{reg} with $m = 1$ is dense in $\mathbb{C}P^{[n_1,\ldots,n_r]}$ and is a genuine manifold. One should note that weighted projective spaces are not in general global orbifolds; that is, they cannot be realised

[9]If F is non-compact, but of course still closed as a subspace of M, one must in general take $\epsilon : F \to \mathbb{R}^+$ to be a positive function on F; the tubular neighbourhood theorem now goes through. Such details will not be important, so we ignore this technicality.

[10]Recall that a map between two G-spaces (spaces with a given action of the group G) is said to be equivariant if it commutes with the group actions. In the case at hand, exp is equivariant since it is defined canonically in terms of the metric, which is G-invariant.

[11]Note also that $\mathbb{C}P^{[1,p]}$ is Thurston's teardrop.

as Y/Γ for some manifold Y and finite group Γ. The reason we mention these facts is that they have been completely overlooked in the physics literature. From the above discussion, we see that only in the case that $n_j \in \{\pm 1\}$ for each j does one obtain a manifold, namely the familiar complex projective space $\mathbb{C}P^{r-1}$. The projection $\mathcal{H} : S^{2r-1} \to \mathbb{C}P^{r-1}$ is then the Hopf map (or antiHopf map, depending on orientation). Thus a necessary condition that B' be a manifold is that all of the n_j, associated with each connected component of $M^{U(1)}$, be equal to ± 1. We therefore assume that the action of Φ is semi-free in the sequel.

4 Brane sources and the Thom class

In this section, we describe more precisely how the stationary point sets in M may be viewed as branes (topological defects) on the base B. We reiterate that the total and reduced spacetimes are $\mathbb{R} \times M$ and $\mathbb{R} \times B$ respectively, but that we deal only with the spatial part of the brane worldvolume in the following. Codimension four stationary point sets in M are of particular interest, since the corresponding branes may be magnetically charged with respect to the Kaluza-Klein two-form, G_2. We derive the corresponding equation (1.1) in the cohomology group $H^3(B)$ using an explicit construction of the Thom class $u \in H^3(E, E_0; \mathbb{Z})$ of the normal bundle E of the brane worldvolume W in the base B [12], where E_0 denotes the complement of the zero section of E. The brane W provides a source for the Kaluza-Klein field strength G_2. Specifically, the equation (1.1) implies that there is a Wess-Zumino source term (1.2) present in the the Kaluza-Klein reduced action[13].This contribution to the action is familiar for example in String Theory where the perturbative critical dimension is $D - 1 = 10$. In this case C_7 is a RR-form potential under which the D6-brane is charged [23].

4.1 Branes as stationary point sets

Let us recapitulate our general setup. (M, g) is an oriented Riemannian manifold, admitting a smooth semi-free orientation-preserving isometric circle action. Let F denote a codimension $2r$ connected stationary point set, which is necessarily closed as a subspace of M. In order to form the quotient space B, we first remove an open invariant tubular neighbourhood NF^ϵ around F, yielding the space M'. The limit in which the radius of this neighbourhood goes to zero corresponds to just removing the stationary points F. If F is compact, the radius may be taken to be $\epsilon > 0$ constant. Otherwise, one may have to take the radius to be a function on F; $\epsilon : F \to \mathbb{R}^+$. The frontier of the tubular neighbourhood is a $(2r-1)$-sphere bundle over F, which is a deformation retraction of the complement of the zero section NF_0 of the normal bundle NF of F in M. The circle action simply corresponds to moving along the fibres of the Hopf (or anti-Hopf) fibration of the $(2r-1)$-sphere, where we have identified NF^ϵ equivariantly with a tubular neighbourhood of F in M via the exponential map.

Now, the image of this frontier in the base $B' = M'/U(1)$ is a $\mathbb{C}P^{r-1}$ bundle over F. Now we come to an important point. This boundary may be interpreted as a brane in B'. The case $r = 2$ is special since the boundary in B' is an $S^2 = \mathbb{C}P^1$ bundle over F, which we may 'fill in' by glueing it to the boundary of an appropriate oriented closed disc bundle over F. More precisely,

[12] note that W is not currently part of the base B; we shall correct this momentarily.

[13] which should be in the 'string frame', as explained later.

the transition functions for this closed disc bundle are given by the transition functions of the two-sphere bundle boundary, so that the latter may be regarded as the boundary of the disc bundle over F. The zero section is interpreted as the brane worldvolume W, and is diffeomorphic to F (although we use different names, to distinguish logically between the submanifold F of M and the image W in the base B). We call the resulting space B, which now has no boundary associated with F. We also have a projection $\pi : M \to B$ where the image of F is W, and the restriction to M' is a $U(1)$ bundle. This construction *only* works for codimension four. The simple reason is that a $\mathbb{C}P^{r-1}$ bundle over some space W is a deformation retraction of the complement of the zero section of a vector bundle over W only if $r = 2$ (the case $r = 1$ is rather trivial as far as we are concerned)[14]. Euclidean space foliates into spheres, not complex projective spaces. The case at hand, however, is degenerate since $S^2 = \mathbb{C}P^1$. This is interesting, since branes in B can be magnetically charged with respect to the Kaluza-Klein two-form G_2 precisely in this dimension. We now show that this is indeed the case. The construction above leads to the interpretation of W as a magnetically charged brane in the base B, satisfying equation (1.1). However, before proving this, we first remind the reader of some facts about branes.

4.2 Brane sources

Let us recall the general theory of a $(p-1)$-form potential C_{p-1}, with field strength $G_p = dC_{p-1}$, on a $(D-1)$-dimensional spacetime $Y = \mathbb{R} \times B$, with $\mathbb{R}$ a trivial time direction. We assume that B contains codimension (p+1) branes W that are magnetically charged with respect to C_{p-1}. Specifically, one has a Wess-Zumino source term

$$\pm(-1)^{D-p} \int_{\mathbb{R}\times W} C_{D-p-2} \tag{4.1}$$

with a + for branes and a $-$ for antibranes, together with the usual bulk action which should be in the 'string' frame

$$-\frac{1}{2} \int_Y *G_p \wedge G_p \tag{4.2}$$

where C_{D-p-2} is the potential for the Hodge dual field strength $dC_{D-p-2} = G_{D-p-1} = *G_p$, the Hodge dual being that defined by the *Lorentz* metric on $\mathbb{R} \times B$ (for example, just take a trivial metric product. The details are not too important). Varying this action leads to the equation in cohomology

$$[dG_p] = \pm[\delta(W)] \tag{4.3}$$

where $\delta(W)$ is a closed $(p+1)$-form that is the Poincaré dual of W in B, and whose support is limited to W, and $[\ldots]$ denotes the image in $H^*(B;\mathbb{R})$.

The equation (4.3) follows from demanding stationarity of the action under the variation $C_{D-p-2} \to C_{D-p-2} + \delta C_{D-p-2}$, such that the cohomology class of G_{D-p-1} is preserved. Note that the variation of the Wess-Zumino term gives

[14]In fact, $\mathbb{C}P^{2n}$ is not the oriented boundary of any oriented $(4n+1)$-manifold with boundary, for $n \geq 1$. We shall have more to say on cobordism in Section 5.

$$\pm(-1)^{D-p}\int_{\mathbb{R}\times W}\delta C_{D-p-2} \tag{4.4}$$

Since δC_{D-p-2} is closed, we may rewrite the integral as

$$\int_{\mathbb{R}\times W}\delta C_{D-p-2}=\int_Y \delta C_{D-p-2}\wedge\eta_{p+1} \tag{4.5}$$

where η_{p+1} is a closed $(p+1)$-form whose cohomology class is Poincaré dual to W in B (see, for example, [24]).

Varying the whole action therefore gives

$$-\frac{1}{2}\delta(*G_p\wedge G_p)+\delta(\pm(-1)^{D-p}C_{D-p-2}\wedge\eta_{p+1})=0 \tag{4.6}$$

and so

$$-\frac{1}{2}\delta(dC_{D-p-2})\wedge G_p-\frac{1}{2}(-1)^{p(D-p-1)+1}*G_p\wedge*\delta(dC_{D-p-2})+\pm(-1)^{D-p}\delta C_{D-p-2}\wedge\eta_{p+1}=0 \tag{4.7}$$

We have used $*^2=(-1)^{p(D-p-1)+1}$ on p-forms. Since d commutes with δ, we have

$$\delta C_{D-p-2}\wedge dG_p=\pm\delta C_{D-p-2}\wedge\eta_{p+1}+d\lambda \tag{4.8}$$

where λ is a global[15] $(D-2)$-form on Y. Hence, taking the image in cohomology, and noting[16] that the support of the Poincaré dual $[\eta]$ of a closed submanifold W may be shrunk into any tubular neighbourhood of W, we finally obtain equation (4.3), in the formal limit that the support is shrunk onto W yielding a delta-function $\delta(W)$. For a general set of branes $\{W\}$ and antibranes $\{\bar{W}\}$, one may extend the above argument linearly to yield

$$[dG_p]=\sum[\delta(W)]-\sum[\delta(\bar{W})] \tag{4.9}$$

as an equation in $H^{p+1}(B)$. If W is compact, the Poincaré dual of W has compact support. Provided that the variation δC_{D-p-2} is assumed to have compact support, we see that equation (4.3) also holds in the compactly supported cohomology group $H_c^{p+1}(B)$, defined in the next section. This will be important for our definition of brane charge. Note that we made the somewhat peculiar choice of coupling $(-1)^{D-p}\int_{\mathbb{R}\times W}C_{D-p-2}$ precisely in order that equation (4.3) has no dependence on dimension. This is natural, as we shall now show from the Kaluza-Klein perspective.

4.3 The Thom class

We now derive equation (4.3) from our Kaluza-Klein perspective, where $p=2$. In this case, G_2 is the Kaluza-Klein field strength. Appropriately normalised, its image in cohomology is therefore the first Chern class $c_1(M')\in H^2(B';\mathbb{Z})$. We assume that the (not necessarily

[15] This word is crucial. Although C_{D-p-2} is not a global form in general, its variation δC_{D-p-2} *is* global.

[16] This is refered to as the Localisation Principal.

connected) stationary point set $M^{U(1)}$ yields a configuration of branes $\{W\}$ and antibranes $\{\bar{W}\}$, arising from each connected component F of $M^{U(1)}$ via the construction in Section (4.1). More specifically, recall that the normal bundle of F in M is a rank four oriented vector bundle, its orientation being the canonical one[17]. Then, as in Section 3, for a semi-free circle action, the induced action on NF is characterised by two integers $n_1, n_2 \in \{\pm 1\}$. If $n_1 = n_2$ then we interpret F as a brane, otherwise F is an antibrane. We could of course have chosen this the other way around, or, alternatively, we could change the orientation of NF. The point is that the choice is unphysical. Interchanging branes with antibranes is a symmetry of the theory; only the relative sign is important. We denote the disjoint sum of branes and antibranes in B as $\mathcal{W} = \mathcal{W}^+ + \mathcal{W}^-$, separating $\mathcal{W}$ into its brane and antibrane constituents.

Let $p : E = N\mathcal{W} \to \mathcal{W}$ denote the normal bundle of $\mathcal{W}$ in the *base* B. This is a rank three orientable vector bundle over $\mathcal{W}$. The Hopf map $\mathcal{H} : S^3 \to S^2$ has first Chern class corresponding to $1 \in \mathbb{Z} \cong H^2(S^2;\mathbb{Z})$, whereas the antiHopf map corresponds to -1. Thus one has a brane $W \subset \mathcal{W}^+$ or an antibrane $\bar{W} \subset \mathcal{W}^-$ depending on the sign of G_2. Note that flipping the sign of G_2 flips the sign of the coupling in (1.2), changing a brane into an antibrane.

In order to derive (4.3), we must introduce the notion of the Thom class of an oriented vector bundle. However, before doing this, we first remind the reader of some definitions. We have assumed so far that the reader is familiar with cohomology groups. For de Rham cohomology, these are roughly speaking the space of closed forms modulo the space of exact forms. However, on a non-compact manifold, one may also define cohomology with compact support. In this case, the cohomology groups are the space of closed forms with compact support modulo the space of exact forms $d\omega$, where ω has compact support. Thus, given a closed compactly supported p-form ν on some manifold B, we may consider ν as an element of both $H^p(B)$ and $H^p_c(B)$. If $\nu = d\omega$ for some global form ω then ν is trivial as an element of $H^p(B)$, but it may not be trivial as an element of $H^p_c(B)$ since ω may not have compact support. This will be important in the following. Finally, for forms defined on the total space of some vector bundle E, we have the notion of cohomology with compact support in the vertical direction; in other words, the forms above need not have compact support on E, but instead the restriction to each fibre is required to have compact support. The cohomology is denoted $H^*_{\mathrm{cv}}(E)$. Similar definitions exist for the singular theory and may be found in Appendix A of [11].

Let $\pi : E \to B$ be a rank n oriented vector bundle over B. Then the cohomology group $H^n(E, E_0;\mathbb{Z})$ contains precisely one cohomology class u, the Thom class, whose restriction $u\mid_{(V,V_0)} \in H^n(V, V_0;\mathbb{Z})$ is the preferred generator given by the orientation of the bundle, for each fibre $V \cong \mathbb{R}^n$, where a subscript 0 denotes the complement of zero (or zero section in the case of bundles). This class enters into the Thom Isomorphism Theorem. This states that

$$\mathcal{T} : H^*(B;\mathbb{Z}) \to H^{*+n}(E, E_0;\mathbb{Z}) \tag{4.10}$$

is an isomorphism, given explicitly by

$$\mathcal{T}(\omega) = (\pi^*\omega) \cup u \tag{4.11}$$

where $\cup$ is the cup product[18]. For a proof of these statements, the reader is again referred to [11].

[17] that is, such that $NF \oplus TF = TM\mid_F$ has the direct sum orientation.

[18] which should of course be replaced by the wedge product $\wedge$ in the de Rham category.

Alternatively, and equivalently, one may replace the relative cohomology $H^*(E, E_0)$ with cohomology with compact support in the vertical direction, $H^*_{\mathrm{cv}}(E)$. Then the Thom class, as above, is uniquely characterised as the cohomology class in $H^n_{\mathrm{cv}}(E)$ which restricts to the preferred generator of $H^n_{\mathrm{c}}(V)$ on each fibre V, where $H^*_{\mathrm{c}}(V)$ denotes the cohomology ring of V with compact support. It is this notion of the Thom class that we shall need.

The first point to note is that the Poincaré dual of a closed oriented submanifold $\mathcal{W}$ of B and the Thom class of the normal bundle of $\mathcal{W}$ in B can be represented by the same forms. Thus if $u \in H^3_{\mathrm{cv}}(T)$ is the Thom class of a tubular neighbourhood[19] T of $\mathcal{W}$ in B, and η is the Poincaré dual of $\mathcal{W}$, we have

$$\eta = j_* u \tag{4.12}$$

as a relation in $H^3(B)$, where j_* denotes extension of u by zero. The proof is straightforward; one merely shows that $j_* u$ satisfies the defining equation of the Poincaré dual.

Alternatively, in terms of the relative theory, since T and $B - \mathcal{W}$ have union B and intersection $T - \mathcal{W}$, one has an excision isomorphism[20]

$$i^* : H^*(B, B - \mathcal{W}) \to H^*(T, T - \mathcal{W}) \tag{4.13}$$

where i denotes inclusion. Thus, combining $(i^*)^{-1}$ with the restriction map $H^*(B, B - \mathcal{W}) \to H^*(B)$ maps the Thom class $u \in H^3(T, T - \mathcal{W})$ to the Poincaré dual of $\mathcal{W}$, $\eta \in H^3(B)$.

Going back to our general discussion of the Thom class of an oriented vector bundle E, we now describe an explicit construction for a representative of the Thom class in terms of the global angular form ψ on E_0. This is described in [24]. It is essentially the vector bundle analogue of passing from a generator of $H^{n-1}(S^{n-1})$ to a generator of $H^n_{\mathrm{c}}(\mathbb{R}^n)$. Given an Euclidean vector bundle E, one may define an associated sphere bundle $S(E)$ given by the subbundle consisting of all unit vectors in E. Then one has a deformation retraction $f : E_0 \to S(E)$ of the complement of the zero section of E onto $S(E)$. The global angular form $\psi_S \in \Omega^{n-1}(S)$ on an oriented $(n-1)$-sphere bundle has two defining properties

- Its restriction to each fibre generates the cohomology of the fibre
- $d\psi_S = -\Pi^* e$

where Π is the sphere-bundle projection, and e is the Euler class of the sphere bundle. When S derives from a vector bundle E, one may pull back ψ_S to E_0 via the deformation f; thus $\psi = f^* \psi_S \in \Omega^{n-1}(E_0)$, which we call the global angular form of E_0. It is now a simple matter to prove that the cohomology class

$$u = [d(\rho(r).\psi)]_{\mathrm{cv}} \in H^n_{\mathrm{cv}}(E) \tag{4.14}$$

is the Thom class of E, where r is the radius (defined by the metric) and all that is required of the function $\rho(r)$ is that ρ be smooth and equal to -1 in a neighbourhood of 0, and equal to

[19] By definition, the exponential map is a homeomorphism onto its image here, and so the tubular neighbourhoods around each connected component of $\mathcal{W}$ are non-intersecting.

[20] See, for example, [12]. Roughly speaking, the idea is that excising simplexes in $B - T$ from both B and $B - W$ does not alter the (co)homology.

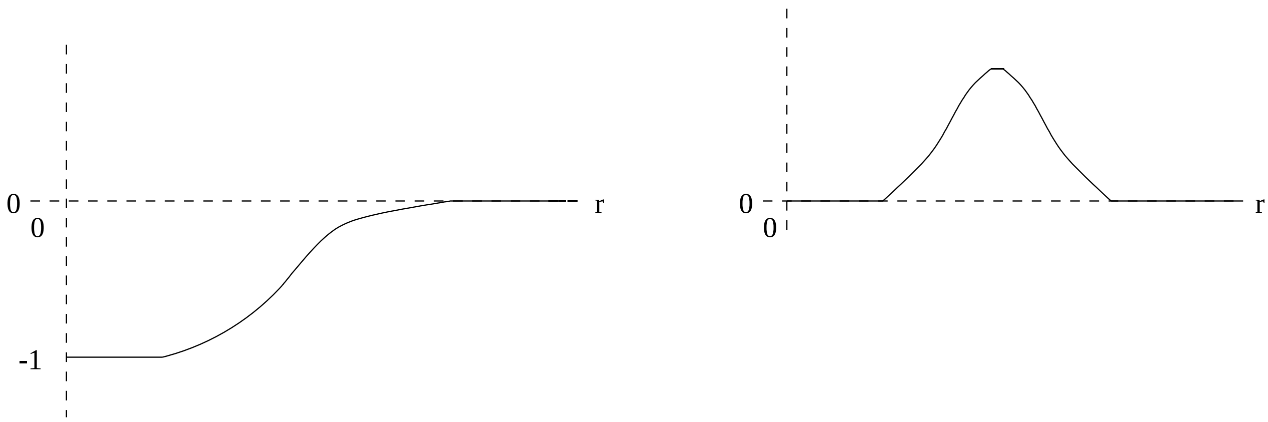

Figure 2: *Left*: The function $\rho(r)$. *Right*: its derivative $\rho'(r)$ is a bump function with total integral 1.

0 at infinity. Its derivative $\rho'(r)$ is then typically a bump function on $\{r \in \mathbb{R} \mid r \geq 0\}$ with total integral 1. Although ψ is defined only on E_0, we have $d\rho \equiv 0$ in a neighbourhood of 0, from which it follows that $d(\rho(r).\psi)$ is a global form on E. One must show that this form is closed (which is trivial), has compact support in the vertical direction, and has unit integral along each fibre. The last two conditions follow from the defining properties of $\rho(r)$ and ψ. Also, it is easy to see that any other function $\tilde{\rho}$ which has the same defining properties as ρ above yields the same cohomology class for u. This completes our description of the Thom class in terms of the global angular form.

After this brief summary, we now turn back to our main discussion. Consider our tubular neighbourhood T of $\mathcal{W}$ in B. Since its rank is odd-dimensional, the Euler class vanishes[21], and hence the global angular form ψ of each connected component of T_0 is closed, $d\psi = 0$. Our first task is to show that $\psi = \pm G_2$, depending on whether the connected component corresponds to a brane or an antibrane, where recall that we normalise G_2 such that its image in cohomology generates the first Chern class $c_1(M')$, and we take $M' = M - M^{U(1)}$. Note first that G_2 is indeed closed on T_0. Since two forms are equal if and only if they are equal locally, we may reduce the problem to a local one. Lift T to its image in M; this is an invariant tubular neighbourhood of $M^{U(1)}$ in M and has rank 4. Now use the metric on M to define normal coordinates on a connected component of this bundle. Specifically, in the case of a brane, in normal coordinates we have

$$ds^2 = \left[dr^2 + \frac{r^2}{4}\left((d\alpha - \cos\beta d\gamma)^2 + d\beta^2 + \sin^2\beta d\gamma^2\right)\right] + dx_1^2 + \ldots + dx_{d-3}^2 \qquad (4.15)$$

The piece in square brackets is the metric restricted to the $\mathbb{R}^4$ fibres, and the remaining piece is the restriction of the metric to an open set $U \subset F$. (α, β, γ) are Euler angles on the unit S^3. The circle action simply rotates around the fibres of the Hopf fibration of the three-spheres, which are themselves fibrered over F. It follows that, in this coordinate system,

[21]Proof: any odd-dimensional vector bundle admits an orientation-reversing automorphism, and e changes sign under orientation reversal.

$$G_2 = \frac{1}{4\pi} d(-\cos\beta d\gamma) = \frac{1}{4\pi} \sin\beta d\beta \wedge d\gamma \tag{4.16}$$

This indeed integrates to 1 on each S^2 fibre, which is as expected, since the image in cohomology of G_2 restricted to an S^2 fibre is $c_1(\mathcal{H} : S^3 \to S^2) \in H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$ which is $1 \in \mathbb{Z}$ for the Hopf bundle. Hence G_2 generates the cohomology of the unit two-sphere fibres. For the case of an antibrane $\bar{W}$, one has the antiHopf, rather than Hopf, action. Hence the cohomology class of G_2 in $H^2(S^2; \mathbb{Z})$ is -1, and $-G_2$ now generates the cohomology of the fibres. It follows that $\psi = \pm G_2$ holds in a tubular neighbourhood of a brane or antibrane, respectively.

For the sake of exposition, we focus on the case of a single compact brane W. The more general case of an arbitrary set of compact branes and antibranes follows in a straightforward manner. Note that G_2 is not currently defined on the whole space B. In order to prove equation (4.3) as a relation in $H^3_c(B)$, we first construct an extension of G_2 on the whole of B, and then show that (4.3) is independent of the choice of extension.

In T_0 we have

$$0 = dG_2 = d(1.G_2) = -d(\rho(r).G_2) + d((1 + \rho(r)).G_2) \tag{4.17}$$

where $\rho(r)$ is the function defined above, with the additional requirement that $\rho \equiv 0$ in a neighbourhood of the boundary of T. It follows that $d(\rho(r).G_2)$ and $d\nu$ are equal as forms on T_0, where $\nu = (1 + \rho(r)).G_2$. Now, since $G_2 = \psi$ for a brane, it follows that the first term on the right hand side of (4.17) is a representative of the Thom class u of the normal bundle of W in B. As we remarked earlier, this is a global form on T. Similarly, ν is a global form on T since the function $(1 + \rho(r))$ vanishes in a neighbourhood of $r = 0$, and therefore ν vanishes in a neighbourhood of the zero section. Thus, since both forms are smooth and equal on T_0, we deduce that they are equal on T also (indeed, both forms vanish on the zero section). Taking the limit in which the support of $\rho(r)$ is shrunk to the origin $r = 0$, we have $\nu = G_2$ on T_0. Thus, extending the definition of G_2 to ν on the whole of T, and taking the image in $H^3_{\mathrm{cv}}(T)$, we obtain

$$u = [dG_2]_{\mathrm{cv}} \tag{4.18}$$

Extending the Thom class by zero outside T, we deduce that for W compact

$$[dG_2]_c = [\delta(W)]_c \tag{4.19}$$

holds as a relation in $H^3_c(B)$, and therefore as a relation in $H^3(B)$. Having proven existence of an extension of G_2 such that (4.19) holds, we now prove uniqueness. The definition of the form ν, the extension of G_2, is certainly not unique. However, suppose that $\eta \in \Omega^2(T)$ is another global extension. Then, by definition, $\eta \equiv G_2$ in a neighbourhood of the boundary of T. Thus the form $\eta - \nu$ has compact support in the vertical direction. Moreover, since η is assumed to be a global form on T, it follows that $\eta - \nu$ is a global form on T, and thus $\eta - \nu \in \Omega^2_{\mathrm{cv}}(T)$. Hence the cohomology class of $d(\eta - \nu)$ in $H^3_{\mathrm{cv}}(T)$ is trivial. This completes the proof. Note, in particular, that the choice of extension of G_2 in (4.19) is independent of the choice of function $\rho(r)$. We shall analyse equation (4.19) in the context of brane charge in the next section.

For an antibrane, one merely inserts $G_2 = -\psi$ into (4.17), yielding a minus sign in (4.19). Hence, for a general configuration of branes and antibranes $\mathcal{W}$, we recover equation (4.9) for the case $p = 2$.

This is precisely what we wanted to show. Let us follow through the logic. A codimension four stationary point set in a $(d+1)$-dimensional manifold M equipped with a semi-free circle action may be regarded, via the construction in section (4.1), as a codimension three brane W in the base d-manifold. If G_2 is the Kaluza-Klein field strength, then this brane acts as a source for G_2 via equation (4.19). Going over to the general theory of branes, this implies that the brane must couple to the dual potential as $(-1)^D \int_{\mathbb{R}\times W} C_{D-4}$; hence one must include this together with the usual Kaluza-Klein reduced action in order to reproduce the correct equation for G_2 in the presence of the brane.

In conclusion, codimension four stationary point sets of semi-free circle actions may be interpreted, upon Kaluza-Klein reduction, as branes or antibranes in the base space that are magnetically charged with respect to the Kaluza-Klein two-form. We have thus both formalised and generalised the ideas in papers such as [1].

4.4 Brane charge

For completeness, we describe the interpretation of equation (4.9) in terms of brane charge, as measured at infinity. This discussion largely follows [26], although we use the examples in Section 2 to illustrate the ideas, and also relate this definition of brane charge to the usual definition in terms of integrals of G_2 over two-spheres. The cohomological definition of brane charge has considerable advantage over the latter definition in that it unambiguously defines the total brane charge of any configuration.

Equation (4.9) states that the total brane charge, defined to be the right hand side, is necessarily zero in the cohomology group $H^{p+1}(B;\mathbb{Z})$, since G_p is required to be globally defined. In particular, if B is compact, this is the familiar statement that the total charge of an abelian gauge symmetry must be zero on a compact manifold. However, in the case that B has non-empty boundary ∂B 'at infinity', and W is compact, one may use the compact Poincaré dual $[\delta(W)]_{\mathrm{c}} \in H^{p+1}_{\mathrm{c}}(B;\mathbb{Z})$, as opposed to the closed Poincaré dual $[\delta(W)] \in H^{p+1}(B;\mathbb{Z})$ that we have been using in most of the discussion so far[22]. There is of course no distinction between the two when B is compact. Although $[\delta(W)]$ is trivial in $H^{p+1}(B;\mathbb{Z})$, it is not necessarily true that $[\delta(W)]_c$ is trivial as an element of $H^{p+1}_{\mathrm{c}}(B;\mathbb{Z})$, the reason being that G_p need not have compact support. The interpretation given in [26] is therefore that brane charge should be interpreted as an element of $H^{p+1}_{\mathrm{c}}(B;\mathbb{Z})$ whose image in $H^{p+1}(B;\mathbb{Z})$ is trivial under the 'forgetful map' $f : H^{p+1}_{\mathrm{c}}(B;\mathbb{Z}) \to H^{p+1}(B;\mathbb{Z})$, that 'forgets' that a class has compact support.

In order to interpret this definition of brane charge in terms of fields measured at infinity ∂B, one may use the exact cohomology sequence for the pair $(B, \partial B)$ [12]

$$\ldots H^p(B;\mathbb{Z}) \longrightarrow^{i^*} H^p(\partial B;\mathbb{Z}) \longrightarrow^{\delta^*} H^{p+1}(B,\partial B;\mathbb{Z}) \longrightarrow^{j^*} H^{p+1}(B;\mathbb{Z}) \longrightarrow \ldots \tag{4.20}$$

where $i : \partial B \to B$ is the inclusion map, and $j : C(B) \to C(B, \partial B)$ is the quotient chain map, and, as earlier, the relative cohomology group $H^{p+1}(B, \partial B;\mathbb{Z})$ is the same as the compactly

[22]For a detailed account, see [24].

supported cohomology group $H_c^{p+1}(B;\mathbb{Z})$ in this case. Using exactness of the long cohomology sequence, we thus conclude that $\ker(f) = H^p(\partial B;\mathbb{Z})/i^*(H^p(B;\mathbb{Z}))$, and so we may interpret brane charge in terms of a field strength G_p on ∂B that does not extend over B as a *closed* form, and therefore must have been created by branes.

In the Kaluza-Klein case, note that if G_2 has compact support, then G_2 vanishes in a neighbourhood of the boundary ∂B. By the classification theorem, this implies that the circle bundle is trivial in a neighbourhood of infinity. We conclude that the configuration has zero brane charge.

We now use the examples in Section 2 to illustrate these rather abstract topological ideas.

The monopole

In this case, the base $B = \mathbb{R}^3$, and the monopole worldline W is just a point $x \in \mathbb{R}^3$, which we may take to be the origin. Now $H^3(\mathbb{R}^3;\mathbb{Z}) \cong 0$, and we may take any three-form on $\mathbb{R}^3$ as the closed Poincaré dual of x. Shrinking the support of this three-form into a neighbourhood of x gives us the unit-integral bump form

$$f(x^1, x^2, x^3)dx^1 \wedge dx^2 \wedge dx^3 \tag{4.21}$$

where (x^1, x^2, x^3) are canonical coordinates on $\mathbb{R}^3$ and f is a bump function on $\mathbb{R}^3$ with support in a neighbourhood of x. In the limit in which we shrink the support to be only at the point x, the function f becomes a Dirac delta function, $\delta(x)$, with support at x.

However, the compactly supported cohomology $H_c^3(\mathbb{R}^3;\mathbb{Z}) \cong \mathbb{Z}$ is non-trivial; one may take the unit bump form (4.21) to be one of the generators. Hence the compact Poincaré dual of the point x is non-trivial as an element of $H_c^3(\mathbb{R}^3;\mathbb{Z})$. Above, we interpreted brane charge as an element of $H_c^3(\mathbb{R}^3;\mathbb{Z})$ that is trivial when mapped to $H^3(\mathbb{R}^3;\mathbb{Z})$ under the forgetful map. In this case, the Kaluza-Klein monopole charge is precisely the unit bump form (4.21) that generates $H_c^3(\mathbb{R}^3;\mathbb{Z})$, since any three-form is trivial when regarded as an element of $H^3(\mathbb{R}^3;\mathbb{Z})$. Having identified the monopole charge with one choice of generator of the compactly supported cohomology, the antimonopole charge corresponds to the other choice of generator.

The D6-brane

The discussion for the D6-brane is straightforward. We first wrap the directions transverse to the brane in order that the worldvolume W be compact. The solution therefore takes the form (2.6) but with $x_1, \ldots, x_6$ periodically identified. W is now a six-torus, T^6, located at $\{r = a\}$ in the Taub-NUT part of the spatial section M. The normal bundle of W in M is just $\mathbb{R}^3$, and therefore the discussion of charge reduces precisely to the case of the monopole above. That is, the $D6$-brane charge[23] corresponds to a choice of generator of $H_c^3(\mathbb{R}^3;\mathbb{Z}) \cong \mathbb{Z}$; the other choice of generator is then associated with the charge of the anti-D6-brane, $\bar{D}6$.

4.5 Relation to the usual definition of brane charge

One would normally define the magnetic charge of a Kaluza-Klein brane W to be

$$Q = \int_{S^2} G_2 \tag{4.22}$$

[23]modulo K-theoretic considerations

where S^2 is a two-sphere that surrounds the brane worldvolume W. One may derive this from our discussion above by evaluating equation (4.18) on a typical fibre V of a tubular neighbourhood of W in B. Thus, since the Thom class has unit integral on each fibre, we deduce that

$$Q = Qu[V] = [dG_2]_{\mathrm{cv}}[V] = \int_V dG_2 = \int_{\partial V} G_2 \tag{4.23}$$

where ∂V is a two-sphere that bounds the fibre V, and we have reinstated the factor of Q in (4.18). However, the advantage of the cohomological definition of brane charge is that it gives an unambiguous definition of the *total* brane charge of an arbitrary configuration. The problem is that one cannot define the total brane charge of an arbitrary configuration as an integral of the Kaluza-Klein two-form over the sphere at infinity, since in general dimension, the sphere at infinity is not a two-sphere. Moreover, there is no natural definition of a two-sphere at infinity in general. One can only use this naive definition of brane charge when the base has dimension three (for example, the monopole). In this case, the cohomological and naive definitions agree on evaluating the former on the fundamental homology class. Similar remarks apply to p-brane charge in general.

5 Kaluza-Klein monopoles

5.1 The general setup

In this section, we study monopole-antimonopole production in a five-dimensional *Euclidean* Kaluza-Klein theory, specialising the above discussion to the case $d = 4$. In dimension four, monopoles are 0-branes that are magnetically charged under the Kaluza-Klein field strength, G_2. The basic static Lorentzian solution is given by (2.1). It turns out that one may use various G-index theorems [10] in order to relate the numbers and types of defects nucleated to the topology of the defects (which is trivial here) and to the topology of M (or rather, a nucleation surface Σ in M).

The setup we consider is the following. Let M be an oriented Riemannian 5-manifold with compact boundary $\partial M = \Sigma$ equipped with a smooth isometric circle action. Since Σ is an oriented boundary, by definition it has trivial oriented cobordism class. By a theorem of Thom [11], the Hirzebruch signature $\mathrm{Sign}(\Sigma)$ of Σ is then necessarily zero. Here, $\mathrm{Sign}(\Sigma)$ is defined as the signature of the non-degenerate quadratic form on $H_2(\Sigma; \mathbb{R})$ defined by the cup-product. It is also given by the index of a certain elliptic operator, associated with the de Rham complex [10] (paper III). It also follows that the Pontrjagin numbers and Stiefel-Whitney numbers of Σ must all vanish [11]. The Euler class of Σ, $\chi(\Sigma)$ is usually defined as the alternating sum of the Betti numbers, b_i, of Σ. Thus $\chi(\Sigma) = \sum_{i=0}^{4} (-1)^i b_i$ where $b_i = \dim(H_i(\Sigma; \mathbb{R}))$. However, it is also the index of the de Rham complex. We will have more to say on this in the next section. Since Σ is a boundary, $\chi(\Sigma)$ is even. To see this, we define the double of M as $2M = M \cup_{\partial M} (-M)$ where $-M$ denotes M with its orientation reversed, and the $\cup_{\partial M}$ symbol denotes that M and $-M$ are to be glued together across ∂M. $2M$ is a closed oriented manifold with a smooth $U(1)$ action. One has the following formula for the Euler characteristic [12]

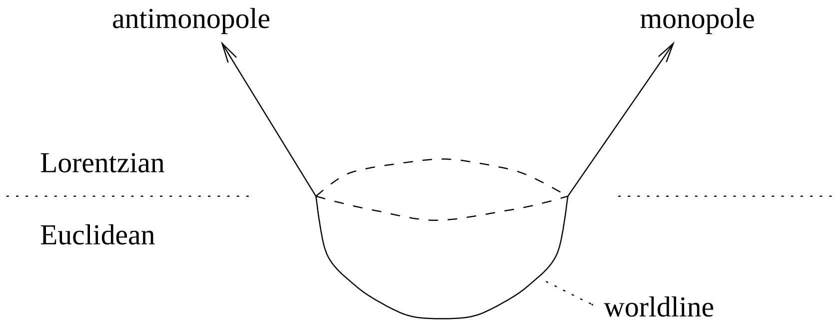

Figure 3: Schematic picture of monopole-antimonopole creation. The dashed line lifts to the nucleation surface Σ, and the solid line is the monopole-antimonopole worldline, which lifts to a geodesic curve in $M \cup_\Sigma M_L$.

$$\chi(\partial M) = 2\chi(M) - \chi(2M) \tag{5.1}$$

Since $2M$ is closed and of odd dimension, $\chi(2M) = 0$ trivially, and thus we see that $\chi(\partial M) = 2\chi(\Sigma) \in 2\mathbb{Z}$. This will be important in the sequel.

We want to interpret Σ as a nucleation surface; that is, M is joined to the post-tunnelling Lorentzian manifold M_L across the totally geodesic (zero-momentum) surface Σ. In M_L Σ is a spacelike boundary, which serves as a Cauchy surface for M_L, and may be interpreted as 'the beginning of time' [21]. The circle isometry of M should extend to a circle subgroup of the isometry group of M_L. We will not consider the Lorentzian sector in detail in the rest of this paper, so we leave out the details of precisely how the analytic continuation is to be performed.

We now suppose that the semi-free circle action on M has a one-dimensional stationary point set, $\dim(M^{U(1)}) = 1$ (note this need not be connected). Thus $\nabla_b k_a$ has rank four. The stationary points then constitute oriented geodesic curves in M. Since k is tangent to Σ, these geodesic curves intersect Σ orthogonally, if at all. In this case a generic connected geodesic curve starts on Σ, continues into M, and then re-intersects Σ at some parameter distance along the curve. These points on Σ are the zeros of $k \mid_\Sigma$ and will correspond to monopoles and antimonopoles (or vice versa, depending on orientation) in the base B, which start life on the nucleation surface $B_\Sigma = \Sigma/U(1)$. The curves themselves are the Euclidean worldlines of the monopoles.

Note that this picture is qualitatively similar to electron-positron pair production in a uniform electric field. In the Euclidean sector, one has a single electron in flat space that travels round in a circle. One may slice this circle along the equator and join onto the post-tunnelling Lorentzian solution, in which one has an accelerating electron-positron pair, pulled apart by the electric field. The electron and positron are then viewed as the *same* particle, the positron being viewed as a negatively charged electron travelling backwards in time, tunnelling through the Euclidean sector, and re-emerging as an electron now travelling forward in time. The Kaluza-Klein monopole picture above is very similar, only here the worldline is identified with a stationary geodesic γ in $M \cup_\Sigma M_L$ and the charge is determined by the linking number $\mathrm{link}(\gamma, \Sigma)$ of this curve with the nucleation surface[24].

Note that from this analysis we see that the number of monopoles will always be equal to the number of antimonopoles; we will be able to prove this later using the G-signature theorem, which refers only to data on Σ. Note that one may also have closed geodesic curves in M. These presumably correspond to virtual monopole-antimonopole loops.

5.2 The G-index theorem

In this section we shall use the G-index theorem [10] to show that the number of monopoles is always equal to the number of antimonopoles (charge conservation), and that the total number of defects produced is given by the Euler number of Σ. Note that these statements are consistent with the fact that $\chi(\Sigma) \in 2\mathbb{Z}$.

The G-index theorem is essentially a generalisation of the Lefschetz fixed point theorem and the usual index theorem. Let $\mathcal{E}$ be an elliptic complex on the compact manifold Σ, and suppose that the (topologically cyclic[25]) compact Lie group G acts on $\mathcal{E}$. Then recall that the Lefschetz number $L(g,\mathcal{E})$ for a generator $g \in G$ of G is defined as

$$L(g,\mathcal{E}) = \sum (-1)^i \mathrm{Tr}(g \mid H^i(\mathcal{E})) \tag{5.2}$$

where $H^i(\mathcal{E})$ are the homology groups of the complex $\mathcal{E}$. The G-index theorem of [10] expresses the Lefschetz number in terms of the symbol of $\mathcal{E}$ and various characteristic classes, evaluated over the fixed point set of g. The general formula is rather complicated. The interested reader is referred to the original paper [10].

We will need to consider two 'classical' complexes; the de Rham complex, and the signature complex. If G acts on Σ, then it acts on the latter complexes. We are of course interested in applying this to $G = U(1)$. Since $U(1)$ is connected, it acts trivially on the cohomology of Σ, and hence in this case the Lefschetz number (5.2) reduces to the usual index. The G-index theorem thus expresses the Euler number and signature of Σ, being the indices of the de Rham and signature complexes respectively, in terms of various characteristic classes evaluated on the $U(1)$ stationary point sets. This is actually reasonably straightforward. The details may be found in the original papers [10], but see also [14]. We simply state the results. For the de Rham complex one obtains

$$\chi(\Sigma) = \sum_F \chi(F) \tag{5.3}$$

where the sum is over each connected component F of the stationary point set $\Sigma^{U(1)}$.

The theorem for the signature complex is rather more involved. However, it simplifies when the stationary points are isolated. If Σ has dimension $2r$ then one easily obtains

$$\mathrm{Sign}(\Sigma) = \sum \prod_{j=1}^{r} (-i) \cot\left(\frac{n_j \tau}{2}\right) \tag{5.4}$$

[24]The linking number is defined to be $+1$ or -1 depending on whether T_pM has the direct sum orientation of $T_p\gamma \oplus T_p\Sigma$ or not, where $p = \gamma \cap \Sigma$ [13].

[25]G is topologically cyclic iff it contains an element $g \in G$ such that the powers of g are dense in G.

The sum is over each isolated fixed point, with $\{n_j\}$ characterising the circle action on the normal bundle (which of course is just the tangent bundle of Σ restricted to the point) as in Section 3, and the formula is valid for *all* values of the group parameter τ.

We now apply this to the case where Σ has $\dim(\Sigma) = 4$ and trivial oriented cobordism class, so that $\mathrm{Sign}(\Sigma) = 0$, and the $U(1)$ action is semi-free, so that all n_j are ± 1. We choose our orientation convention by defining a monopole fixed point (or 'nut' in the terminology of [15]) to have $\prod_{j=1}^{2} n_j = +1$ and an antimonopole fixed point (or 'antinut') to have $\prod_{j=1}^{2} n_j = -1$. The G-signature theorem thus states that the number of nuts minus the number of antinuts is zero. Hence the number of monopoles is always equal to the number of antimonopoles. This of course agrees with our earlier discussion. There is, however, a slight subtlety in the argument. The index theorem does not assume that Σ bounds a manifold with a smooth circle action, inducing the given circle action on Σ. In principle such an extension might not exist. This point is in any case irrelevant, since the existence of M with boundary Σ is assumed from the outset. Nevertheless, it is interesting that the signature complex is related to charge conservation in this way.

The G-index theorem applied to the de Rham complex (5.3) gives us a less trivial result. Note that the Euler characteristic of an oriented manifold is invariant under a change of orientation. The Euler characteristic of a point is just one, and by our previous remark is independent of how we choose to orient the point. Hence when all stationary points are isolated, (5.3) implies that $\chi(\Sigma)$ is equal to the total number of monopoles and antimonopoles. This completes our analysis of the fixed point theorems.

5.3 Circle actions on 4-manifolds and cobordism

We now give a summary of a rather remarkable result due to Fintuschel [6] on the classification of circle actions on 4-manifolds. We can then apply this classification to our nucleation surface Σ, giving a complete description of the possible topological configurations of monopole-antimonopole nucleation in $D = 5$ Kaluza-Klein theory.

The main result of [6] is that if Σ is a simply-connected 4-manifold admitting a smooth circle action[26], then, modulo the Poincaré conjecture[27], Σ is the connected sum of copies of S^4, $S^2 \times S^2$, $\mathbb{C}P^2$ and $-\mathbb{C}P^2$. This rather deep result will be useful for classifying, topologically, monopole-antimonopole production.

Recall that the connected sum $X \# Y$ of two closed oriented d-manifolds X and Y is defined by removing (sufficiently small) d-balls from each, and then identifying the boundaries of the balls[28]. It is easy to show that $H_d(X \# Y; \mathbb{Z}) \cong \mathbb{Z}$ and

$$H_r(X \# Y; \mathbb{Z}) \cong H_r(X; \mathbb{Z}) \oplus H_r(Y; \mathbb{Z}) \tag{5.5}$$

for $0 < r < d$.

[26] In the original paper, the action is assumed to be *locally smooth.*. However, any smooth action of a compact Lie group is locally smooth [16].

[27] Any 3-manifold that is homotopy-equivalent to S^3 is homeomorphic to S^3. This remains as probably one of the greatest unsolved problems in topology.

[28] One must of course smooth out the resulting space in a neighbourhood of the identification. This can be done.

It is clear that given a manifold X, $S^d\#X$ is diffeomorphic to X. Thus the S^4 contribution in Fintuschel's Theorem is rather trivial.

We may now apply the theorem to our nucleation surface Σ, assuming the latter is simply-connected. In many cases, one might achieve this by going to a suitable covering space, although this may result in a non-compact manifold. T^4 is the obvious example. Since Σ is a boundary, its cobordism class is trivial. Recall that two compact oriented d-manifolds X and Y belong to the same cobordism class [17] if and only if there is a compact oriented $(d+1)$-manifold M with boundary ∂M such that ∂M, with its canonical orientation, is diffeomorphic to $X+(-Y)$, where $+$ denotes the disjoint sum (distinguished from the connected sum $\#$). One may then define an abelian group Ω_d consisting of all oriented cobordism classes of d-manifolds, the group composition being $+$. The low dimensional oriented cobordism groups are conveniently listed in [11]. In particular, $\Omega_4 \cong \mathbb{Z}$, the generator being $\mathbb{C}P^2$. It follows that $\mathbb{C}P^2\#-\mathbb{C}P^2$ has trivial cobordism class (that is, it bounds). In fact, a manifold X is an oriented boundary if and only if all the Pontrjagin numbers[29] and Stiefel-Whitney numbers of X vanish [17]. One may easily verify that this is the case for $\mathbb{C}P^2\#-\mathbb{C}P^2$. Hence one must have equal numbers of $\mathbb{C}P^2$ and $-\mathbb{C}P^2$ in the classification theorem.

One may thus conclude that Σ is either S^4 or the topological sum of $S^2\times S^2$ and $\mathbb{C}P^2\#-\mathbb{C}P^2$. In fact, there is an alternative way to view the latter. Recall that G-bundles over S^n are classified by $\pi_{n-1}(G)$ [18]. In the case $n=2$, we conclude that S^2 bundles over S^2 are classified by $\pi_1(SO(3)) \cong \mathbb{Z}_2$. Hence, up to bundle isomorphism, there are two S^2 bundles over S^2; the trivial bundle $S^2\times S^2$, and precisely one twisted $S^2\times S^2$ product. The latter is in fact diffeomorphic to $\mathbb{C}P^2\#-\mathbb{C}P^2$. This may be seen via the explicit construction of the Page instanton [19] on the twisted S^2 bundle over S^2 in [20]. Thus Σ is S^4 or the connected sum of S^2 bundles over S^2.

If we impose that Σ be a spin manifold, so that the second Stiefel-Whitney class $w_2(\Sigma) \in H^2(\Sigma;\mathbb{Z}_2)$ vanishes, then neither $\mathbb{C}P^2$ nor its orientation-reversed cousin $-\mathbb{C}P^2$ may contribute in the above. Thus, in this case, we have a 1-1 correspondence between the possible topologies of Σ and the Euler number of Σ. Specifically the possible Euler numbers for Σ are $2n$ ($n\in\mathbb{N}$) where $n=1$ corresponds to S^4 and $n>1$ corresponds to the connected sum of $n-1$ copies of $S^2\times S^2$. This is easily computed from the formula (5.5). The Euler number is just the total number of isolated stationary points, fixed under some semi-free circle action, which we may think of as n monopole-antimonopole pairs which start life on the nucleation surface $B_\Sigma = \Sigma/U(1)$. Note, however, that each Σ may admit many inequivalent circle actions, leading to different base spaces B_Σ.

6 Illustrations

As an application of these results, we consider several explicit examples describing the production of Kaluza-Klein monopole-antimonopole pairs. Up until this point, we have largely been concerned with topological issues. In the case that the nucleation surface is both simply-connected and spin, we have seen that the possible topologies are classified uniquely in terms of the total number of monopoles. However, the possible physics described by this topological

[29] In dimension four, one has $\mathrm{Sign}(X) = \frac{1}{3}p_1[X]$ where $p_1 \in H^4(X;\mathbb{Z})$ is the first Pontrjagin class of X.

data is far from unique. In order to produce pairs of objects, one needs some sort of physical mechanism in order to pull the pair apart. For example, an electric field in the case of electron-positron pairs, a positive cosmological constant in the case of black holes, etc. As an illustration, we consider two distinct physical mechanisms that support the production of monopole-antimonopole pairs. In Section (6.1), we examine the case of a positive cosmological constant. As is well known, this is a source of gravitational repulsion, and is therefore able to support the pair-production process. The topology of the nucleation surface is S^4, and hence we produce a single pair of defects, in agreement with our general analysis. In contrast, in Section (6.2) we consider monopole-antimonopole production in the presence of a thin domain wall. The gravitational field produced by a domain wall is repulsive, and this again causes the monopole-antimonopole pair to accelerate apart in the Lorentzian sector. In Section (6.3) we consider a domain wall configuration where the topology of the nucleation surface is again S^4. Moreover, the solution is actually unique for a given domain wall tension, if we demand that the five-dimensional solution is flat either side of the wall [4]. In Section (6.4) we construct a domain wall solution with nucleation surface of topology $S^2 \times S^2$. The total number of monopoles and antimonopoles is 4, again in agreement with our general analysis.

6.1 Cosmological constant

Our starting point is the $(D+1)$-dimensional Euclidean gravitational action[30]

$$I_{\mathrm{EH}} = -\frac{1}{16\pi G_{D+1}} \int_M \mu(g_{D+1}) \left(R(g_{D+1}) - 2\Lambda_{D+1}\right) \tag{6.1}$$

where $R(g_{D+1})$ is the Ricci scalar of the $(D+1)$-metric g_{D+1},$\mu(g_{D+1})$ denotes the canonical Riemannian measure, $\Lambda_{D+1} > 0$ is the cosmological constant, and G_{D+1} is the $(D+1)$-dimensional Newton constant. Locally, one may choose coordinates on M adapted to the Killing vector field k of the circle action Φ. The metric g_{D+1} then takes the local form

$$g_{(D+1),\mu\nu}dx^\mu dx^\nu = e^{\alpha\varphi} g_{D,ij}dx^i dx^j + e^{\beta\varphi}(d\tau + C_1)^2 \tag{6.2}$$

where $0 \leq \tau \leq 2\pi R$ parametrises the circle of radius R, and all fields are independent of τ. The constants α and β are given by $\alpha = \sqrt{\frac{2}{(D-1)(D-2)}}$ and $\beta = -(D-2)\alpha$. Note that if one tries to extend this local coordinate system over a stationary point where $k = 0$, the dilaton φ necessarily diverges there. The action (6.1) reduces to the *Einstein* frame Einstein-Maxwell-dilaton form

$$I_{\mathrm{EH}} = -\frac{1}{16\pi G_D} \int_B \mu(g_D) \left(R(g_D) - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{4}e^{-(D-1)\alpha\varphi}G_2^2 - 2\Lambda_{D+1}e^{\alpha\varphi}\right) \tag{6.3}$$

where $G_2 = dC_1$, $G_D = G_{D+1}/2\pi R$ and the base $B' = \left(M - M^{U(1)}\right)/U(1)$. Notice the potential term $2\Lambda_{D+1}e^{\alpha\varphi}$.

As we have shown in Section 4, the presence of Kaluza-Klein branes in the base B requires that one adds a Wess-Zumino coupling of the form (1.2) to the *string* frame action, in order

[30] M is closed, and note that we have changed our conventions for the total dimension of spacetime (which is now Euclidean).

to reproduce the correct equation for the Kaluza-Klein field strength, (1.1). Thus the total effective action is given by

$$I = I_{\text{String}} + I_{\text{WZ}} \tag{6.4}$$

Of course, $I_{\text{String}} = I_{\text{Einstein}} \equiv I_{\text{EH}}$ numerically; the string frame action is related to the Einstein frame action by a field redefinition.

The maximally symmetric solution to the equations of motion obtained from the action (6.1) is of course the $(D+1)$-dimensional sphere, S^{D+1}, which may also be viewed as the Euclidean version of de Sitter space, dS_{D+1}. The $(D+1)$-sphere metric

$$ds^2_{D+1} = d\psi^2 + \cos^2\psi d\Omega_D \tag{6.5}$$

may therefore be regarded as an instanton for the creation of a $(D+1)$-dimensional universe by a positive cosmological constant, where, without loss of generality, we have set $\Lambda = \frac{1}{2}D(D-1)$ in order to obtain a unit sphere as solution. The tunnelling manifold M is given by $-\frac{\pi}{2} \leq \psi \leq 0$ where $\psi = -\frac{\pi}{2}$ is the South pole of a $(D+1)$-hemisphere, bounded by the equatorial D-sphere $\psi = 0$, which is zero-momentum and therefore may be taken as our nucleation surface Σ. One may obtain the Lorentzian section by analytically continuing $\psi = it$, with $t > 0$ giving the expanding de Sitter solution.

We now specialise to the case $D = 4$. Since $\Sigma = S^4$, it follows that any smooth circle action will produce precisely one monopole-antimonopole pair upon dimensional reduction. Specifically, one may foliate the four-sphere by three-spheres, and take the Hopf action on the latter

$$d\Omega_4^2 = d\rho^2 + \frac{1}{4}\sin^2\rho\left((d\alpha + \cos\beta d\gamma)^2 + d\beta^2 + \sin^2\beta d\gamma^2\right) \tag{6.6}$$

where ρ is a polar coordinate on the four-sphere, $0 \leq \rho \leq \pi$, and (α, β, γ) are Euler angles on the unit three-sphere. The circle action is generated by $\partial/\partial\alpha$ which has isolated fixed points on the North and South poles of the four-sphere, $\rho = 0$, $\rho = \pi$, respectively. These reduce to a monopole and antimonopole on the base upon Kaluza-Klein reduction. The Euclidean worldline is identified with the geodesic curve that starts at the North pole on the equatorial four-sphere, moves back into M, intersecting the South pole of the five-hemisphere at $\psi = -\frac{\pi}{2}$, and continuing back up to the South pole on the equatorial four-sphere. In the Lorentzian sector the monopole and antimonopole accelerate apart. This is illustrated schematically in Figure 3.

6.2 Domain walls

We consider a thin domain wall $Y_{D+1} \subset M$, invariant under the circle action, with Nambo-Goto action

$$I_{\text{DW}} = \sigma_{D+1}\int_{Y_{D+1}} \mu(h_{D+1}) + \frac{1}{8\pi G_{D+1}}\int_{Y_{D+1}} \mu(h_{D+1})[K_{D+1}]_+ \tag{6.7}$$

where σ_{D+1} is the tension of the domain wall with worldvolume Y_{D+1}, and h_{D+1} is the induced metric on Y_{D+1}. Notice that Y_{D+1} and the metric h_{D+1}, being a hypersurface in the $(D+1)$-manifold M, actually have dimension D, and not $D+1$. The label $D+1$ therefore refers to the fact that objects live in the total space M, rather than denoting their actual dimension. We hope this doesn't cause any confusion. We assume that the Killing vector field is non-vanishing on Y_{D+1}. The unit normal n_{D+1} to the domain wall points *into* M on either side of the wall. The second fundamental form of the imbedded hypersurface Y_{D+1} is in general discontinuous across the wall since the metric is continuous, but in general non-differentiable, across Y_{D+1}. We must therefore include a Gibbons-Hawking term in the action, summing the trace K_{D+1} of the second fundamental form of Y_{D+1} on either side of the wall.

Upon dimensional reduction, the domain wall action (6.7) becomes

$$I_{\mathrm{DW}} = \sigma_D \int_{Y_D} \mu(h_D) + \frac{1}{8\pi G_D} \int_{Y_D} \mu(h_D)[K_D + \frac{\alpha}{2} n_D.\partial\varphi]_+ \tag{6.8}$$

where $Y_D = Y_{D+1}/U(1)$ denotes the image of the worldvolume of the invariant domain wall in the base, h_D is the induced metric, $\sigma_D = 2\pi R \sigma_{D+1}$ and n_D is the unit normal (with respect to g_D) of Y_D in B, with trace of the second fundamental form K_D.

We thus see that a domain wall of tension σ_{D+1} in a purely gravitational $(D+1)$-dimensional background may be viewed, upon Kaluza-Klein reduction, as a domain wall of tension σ_D in an Einstein-Maxwell-dilaton background.

Setting $\Lambda_{D+1} = 0$, we see that varying the gravitational action

$$I_{\mathrm{grav}} = I_{\mathrm{EH}} + I_{\mathrm{DW}} \tag{6.9}$$

with respect to the metric yields the Israel matching conditions[31]

$$[K_{(D+1),\mu\nu} - K_{D+1} h_{(D+1),\mu\nu}]_+ = 8\pi G_{D+1} \sigma_{D+1} h_{(D+1),\mu\nu} \tag{6.10}$$

One may find solutions to the equations derived from the action (6.9) as follows. One starts with a $(D+1)$-dimensional Ricci-flat manifold (M, g_{D+1}) admitting a semi-free isometric circle action. One then tries to find a totally umblic invariant hypersurface $Y_{D+1} \subset M$, that is $K_{(D+1),\mu\nu} = c h_{(D+1),\mu\nu}$ for some constant c, that bounds some compact region with boundary Y_{D+1}. One then constructs the double $2M = M \cup_{Y_{D+1}} -M$. The hypersurface Y_{D+1} becomes a domain wall whose tension is determined by the constant c. The Israel equations (6.10) are easily seen to be satisfied provided

$$c = -\frac{4\pi G_{D+1}}{D-1} \sigma_{D+1} = -\frac{4\pi G_D}{D-1} \sigma_D \tag{6.11}$$

This provides us with a simple procedure for constructing domain wall solutions.

Since $(2M, g_{D+1})$ is almost everywhere Ricci-flat, the on-shell gravitational action becomes

$$I_{\mathrm{grav}} = -\frac{\sigma_{D+1}}{D-1} \mathrm{vol}(Y_{D+1}) \tag{6.12}$$

6.3 $\chi(\Sigma) = 2$

In this section, we analyse briefly the solution in [4] and compute its action. Following the above procedure, one starts with five-dimensional flat space $\mathbb{E}^5$

[31] For a nice derivation, see [22].

$$ds_5^2 = dr^2 + r^2\left[d\psi^2 + \frac{1}{4}\cos^2\psi\left((d\alpha + \cos\beta d\gamma)^2 + d\beta^2 + \sin^2\beta d\gamma^2\right)\right] \tag{6.13}$$

The circle action is given by the Hopf action on the S^3 leaves that foliate the S^4 principal orbits of the $SO(5)$ isometry group. That is, we take the Killing vector field $\partial/\partial\alpha$. The invariant umbilic hypersurface is given by $\{r = r_0 > 0\}$ and one easily verifies that

$$r_0 = \frac{3}{4\pi G_5 \sigma_5} = \frac{3}{4\pi G_4 \sigma_4} \tag{6.14}$$

The gravitational action (6.12) is then given by

$$I_{\chi=2} = -\frac{2\pi}{3G_5} r_0^3 \tag{6.15}$$

The double $2M$ is almost everywhere flat and is topologically S^5.

The nucleation surface must be invariant and totally geodesic. One may easily see that $\psi = 0$ satisfies these requirements. The Euclidean tunnelling manifold is then given by $-\frac{\pi}{2} \leq \psi \leq 0$, the upper bound corresponding to the nucleation surface $\Sigma \subset 2M$, which is topologically S^4. The circle action restricted to Σ has two isolated fixed points, in agreement with our general analysis. These are a nut and antinut, separated by the domain wall restricted to the nucleation surface, and are located at the two copies of $r = 0$ either side of the wall. The worldline of the monopole-antimonopole pair is the geodesic curve $\psi = -\frac{\pi}{2}$. This runs from $r = 0$ on the nucleation surface on one side of the wall, intersects the domain wall at $r = r_0$, and then continues to the other copy of $r = 0$ on Σ.

One may obtain the Lorentzian solution by analytically continuing $\psi = it$ with $t > 0$. The monopole-antimonopole pair thus accelerate apart in the Lorentzian section.

6.4 $\chi(\Sigma) = 4$

In this section we construct a solution with two monopole-antimonopole pairs. The nucleation surface, being spin, thus necessarily has the topology $S^2 \times S^2$. One starts with the five-dimensional Schwarzschild solution

$$ds_5^2 = \left(1 - \left(\frac{r_H}{r}\right)^2\right) d\tau^2 + \left(1 - \left(\frac{r_H}{r}\right)^2\right)^{-1} dr^2 + r^2\left[d\psi^2 + \cos^2\psi(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{6.16}$$

Since the coordinate τ is identified with period $2\pi r_H$, we see that the topology is $\mathbb{R}^2 \times S^3$. We take the circle action generated by the Killing vector field $k = \frac{\partial}{\partial\tau} + \frac{1}{r_H}\frac{\partial}{\partial\phi}$. One may easily verify that there exists an invariant umbilic hypersurface $\{r = r_0 > 0\}$ given by $r_0 = \sqrt{2} r_H$, where the tension σ_5 of the resulting domain wall is related to r_0 via

$$\frac{r_0}{\sqrt{2}} = r_H = \frac{3}{8\pi G_5 \sigma_5} = \frac{3}{8\pi G_4 \sigma_4} \tag{6.17}$$

The double $2M$ has topology $S^2 \times S^3$. Note that since the solution (6.16) is regular only if $\tau \sim \tau + 2\pi r_H$, we see that, in this case, the radius of the Kaluza-Klein circle direction is related to the tension of the domain wall. This is in contrast to the previous example, where these quantities were independent. The gravitational action (6.12) is given by

$$I_{\chi=4} = -\frac{\pi^2}{\sqrt{2}G_5} r_0^3 \tag{6.18}$$

The nucleation surface is given by $\psi = 0$, and is topologically $S^2 \times S^2$. The tunnelling manifold is given by $-\frac{\pi}{2} \leq \psi \leq 0$, the upper bound corresponding to Σ. The circle action generated by $k \mid_\Sigma$ has four isolated fixed points at the two copies of the two points $\{r = r_H, \theta = 0\}$ and $\{r = r_H, \theta = \pi\}$. One thus has two monopole-antimonopole pairs separated by the domain wall restricted to Σ. The mirror image of one monopole-antimonopole pair is an antimonopole-monopole pair. The analytic continuation is again given by setting $\psi = it$.

7 Conclusions

The aim of this paper was to make more precise the relationship between Kaluza-Klein branes and stationary points of circle actions. In references [1] and [2], various branes were constructed as stationary point sets of circle actions, but the relationship to the general theory of branes, and in particular the cohomology equation (1.1), was unclear. Moreover, only simple examples were presented. This motivated the work in the present paper. In particular, in Section 4 we have shown quite generally how codimension four stationary point sets may be interpreted as magnetically charged branes in the reduced space, and, using an explicit construction of the Thom class of the normal bundle of the brane in the base, were able to prove the corresponding equation in cohomology (1.1). This puts the results of papers such as [1] in a more general setting. Note, however, that since the dilaton diverges as one approaches the brane, physically the spacetime decompactifies in a neighbourhood of the brane worldvolume. So, strictly speaking, the physics near the brane is goverened by the higher dimensional theory. However, if one wishes to interpret the brane purely from the lower-dimensional point of view, in order to make contact with the general theory of branes, one must resort to a construction similar to the one outlined in this paper.

In Section 5 we then went on to study the specific case of monopole-antimonopole production in a five-dimensional Kaluza-Klein theory. Charge conservation and the number of defects produced are related to various G-index theorems, and using Fintuschel's classification of circle actions on simply-connected four-manifolds, together with some simple cobordism results, we were able to classify completely the possible topologies of the nucleation surface. Finally, in Section 6 we gave several explicit examples of monopole-antimonopole nucleation, where the production mechanism is supported either by a positive cosmological constant, or a domain wall.

In conclusion, the theory of Kaluza-Klein branes is related to many different areas of algebraic and differential topology; G-index theorems, cobordism, and various characteristic classes all play an important role in the general theory. This, together with recent work on the relation of K-Theory to the physics of D-branes, suggests that perhaps there should exist a deeper and more fundamental mathematical structure that underlies all of these ideas.

Acknowledgments

I am extremely grateful to Gary Gibbons for valuable discussions and comments, and would like also to thank Burt Totaro and Stephen Hawking for useful conversations, and Harvey Reall for comments on a preliminary draft.

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